Torus Knots

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Abstract

This is a self-contained paper assuming no background knowledge of knot theory. A knot is a closed curve in three-dimensional space that does not intersect itself anywhere. A (double) crossing is where two strands of the knot meet; a triple crossing is where three strands of the knot meet. In this paper, a specific family of knots, called torus knots, is studied. This paper explores different properties of knots and how they are realized in torus knots. The relationship between the number of double crossings and triple crossings is investigated for a specific type of torus knot.

Contents

1 Introduction

A recent development in knot theory within the last 10 years has been the idea of triple crossings, or where three strands of the knot cross, instead of the traditional two strands. Knot theory is an area of math under the subject of topology; topology is a mathematical subject, just as are algebra or geometry, that explores properties of shapes that remain the same under transformations. This paper will introduce knot theory at a basic level, assuming no prior knowledge of topology or knot theory. The aim of this paper is to prove a result proposed by Colin Adams in [3], which suggests that someone show that 2-braid knots ($(p, 2)$-torus knots) are the only knots with triple crossing number $c(K) - 1$ for a knot K. While the proposed theorem in Section 4 would appear, upon inspection, to follow easily from the proof of Theorem 2.3 in [2], the proof of the theorem in Section 4 is not trivial and relies on two lemmas. In section 2, the necessary background from knot theory is explained and illustrated; this includes some of the simplest knots, methods for manipulating knots, two different notations for knots, and properties of knots. In section 3, the background is given for the specific problem investigated in this paper; as such, section 3 introduces torus knots and explores patterns in the properties of torus knots. Section 4 states the results of the investigated problem. In section 5, the implications of the results are discussed and directions for future research are proposed.

2 Background

In this section, basic terms and notations used in knot theory are defined. The definition of a knot is given, and the three simplest knots are introduced. The methods for doing things to knots and the various notations for knots are explained, and the definition of a link is given. The definition and examples of knot invariants are provided, and finally the braid representation of a knot is explained.

2.1 What is a Knot?

A **knot** is a closed curve in three-dimensional space that does not intersect itself. While a knot does not intersect itself, it will usually cross over or underneath itself when projected onto a plane. Depending on which strand of the knot is being characterized, a crossing is called either an **overcrossing** or an **undercrossing**, according to whether the strand passes over or under another strand. Figure 1 (a) illustrates this. A **strand** is a portion of a knot that goes from one under-

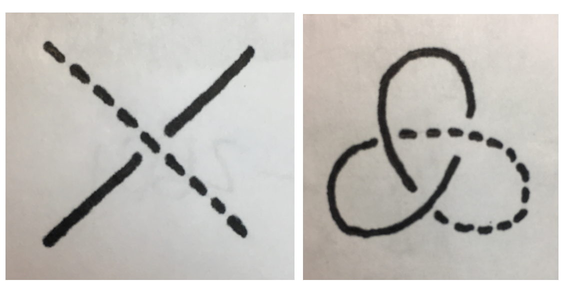


Figure 1: (a) A crossing; the black part is an undercrossing, while the dashed part is an overcrossing. (b) The dashed part of the knot is a strand

crossing to another with only overcrossings in between; in Figure 1 (b), the dashed part is a strand. A **projection** of a knot is a two-dimensional picture of a knot; one knot can have many different projections. Figure 2 gives an example of two projections of the same knot. An **orientation** on a knot is an arbitrarily chosen direction in which one can travel around, or traverse, the knot. Figure 3 shows an oriented knot. Three simple knots to be familiar with are the unknot, the trefoil, and the figure eight knots. The **unknot**, also called the **trivial knot**, is simply an untangled loop; its most common projection is a circle, as shown in Figure 4 (a). The **trefoil knot**, shown in Figure 4

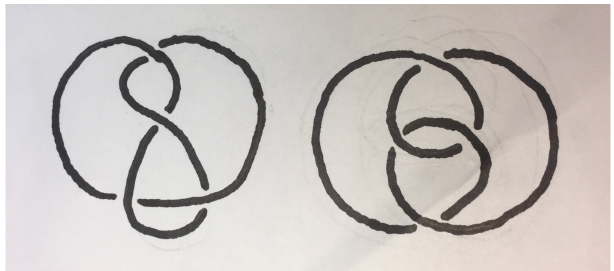


Figure 2: Two Projections of the Figure 8 Knot

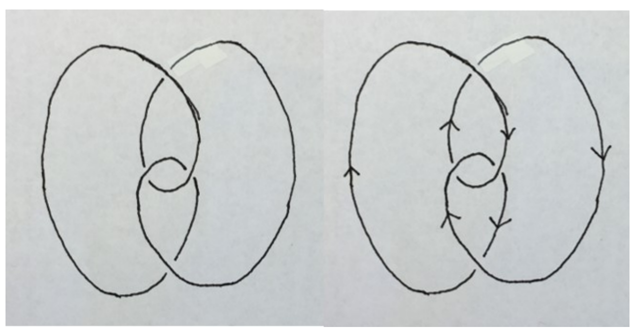


Figure 3: A knot without an orientation, and the same knot with an orientation

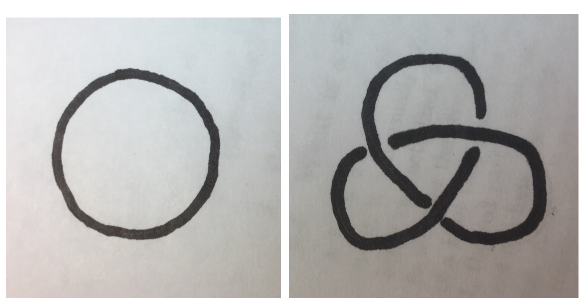


Figure 4: (a) The unknot. (b) The trefoil knot.

(b), is the only nontrivial three-crossing knot and is also called the pretzel knot. The **figure eight knot** is the only nontrivial four-crossing knot, and it is shown in Figure 2.

2.1.1 Knot Manipulations

In order to study knots, a way to manipulate them is needed. An **isotopy** is one such manipulation, as it continuously deforms either the plane or the three-dimensional space that the knot sits in. Specifically, a **planar isotopy** is a deformation of the projection plane as if it were made of rubber with the knot drawn on top, so the knot becomes deformed as well. Figure 5 illustrates a planar isotopy. In contrast, an **ambient isotopy** is a deformation of the three-dimensional space that the knot sits in, moving the knot along with it. No crossings are changed in an isotopy. The main way that knots are manipulated does in fact change the crossings of the knot without fundamentally changing the knot; these are the **Reidemeister moves**, which are shown in Figure 6. A **type I Reidemeister move** allows for a twist in the knot to be created or removed; in other words, it allows for the addition or subtraction of one (trivial) crossing. A **type II Reidemeister move** allows for two crossings to be created or removed from the knot. A **type III Reidemeister move** allows for a strand that is either above or below a crossing to be moved to the other side of the crossing. Two knots are **equivalent** if one can be changed into the other through a series of planar isotopies and Reidemeister moves.

Figure 5: Two planar isotopies [1]

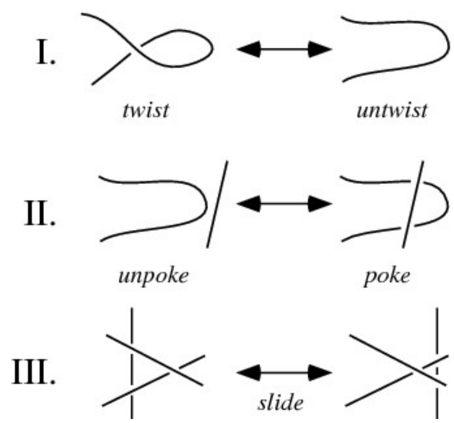


Figure 6: The three types of Reidemeister moves [5]

2.2 Links

A **link** is a set of knotted loops tangled together, and it is often the general term used to speak about properties of both links and knots. A **component** of a link is one of the knotted loops in the link. If a link has two knotted loops, it is said to be a link with two components; if a link has n knotted loops, it is a link with n components. Figure 7 shows two of the simplest links of two components. If a link can be manipulated such that the components are ultimately separated, it is said to be a **splittable link**. The **unlink**, also called the **trivial link**, of n components is the set of n disjoint trivial knots. Figure 8 shows a trivial link with three components.

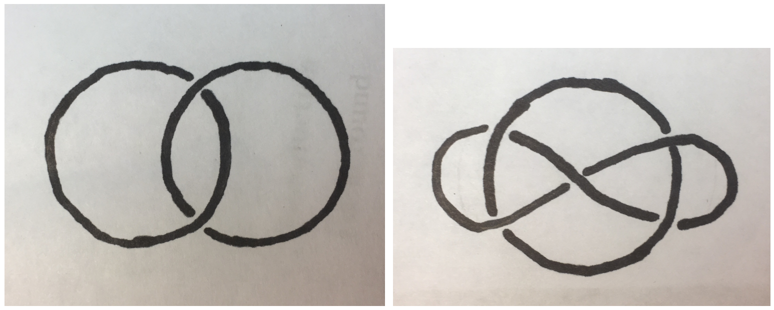


Figure 7: (a) The Hopf Link. (b) The Whitehead Link.

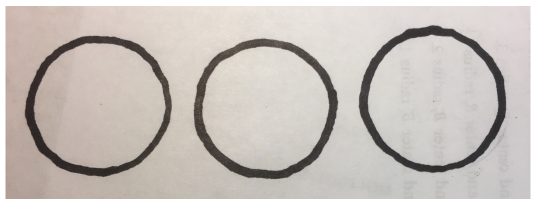


Figure 8: The unlink of 3 components

2.3 Dowker Notation

The Dowker notation of a knot projection is a sequence of even numbers, each of which correspond to a crossing of the knot. To determine the Dowker notation of a knot projection, one begins by giving the projection an orientation. Next, label an undercrossing with a 1, as shown in Figure 9 (a). Starting at that undercrossing and following the orientation of the knot, label each crossing with increasing integers, as in Figure 9 (b). For the even numbers, if the crossing is an overcrossing, leave the even number positive, but if the crossing is an undercrossing, then make the even number negative. This procedure indicates whether a knot is alternating or not; for an alternating knot, since the crossing labeled 1 is an undercrossing, each undercrossing is labeled with an odd number, so each overcrossing is labeled with a positive even number. In a non-alternating knot, there will be either 2 or more consecutive undercrossings or overcrossings, so at least one even number will be negative. Note that odd numbers are always positive. When finished, each crossing will have two numbers labeling it, an even and an odd; the largest number overall will be 2 times the number of crossings in the projection, as shown in Figure 9 (c). Next, off to the side, write down all the odd numbers in increasing order, starting with one, as in Figure 10 (a). Below each odd number, write down the even number that corresponds to the same crossing as the odd number, as in Figure 10 (b). This sequence of even integers is the Dowker notation for the knot projection.

Because the Dowker notation is determined by traversing a knot according to a consistent orientation, it can be used to redraw a projection of the knot by drawing all the crossings in order, as shown in Figure 11. As examples, the Dowker notation of the trefoil is 4 6 2, and the Dowker notation for the figure eight knot is 4 6 8 2. However, the Dowker notation depends on the projection of the knot. Figure 10 (b) depicts this by showing the Dowker notation for a non-standard projection of the figure eight knot; note that it is not the same as 4 6 8 2, as mentioned previously.

2.4 Conway Notation

The Conway notation is more sophisticated and revolves around tangles. A **tangle** in a knot or link projection is an area that can be surrounded by a circle that is crossed by the knot or link exactly four times. Figure 12 gives two examples of tangles. The ∞ tangle is two non-crossing vertical strings, and the 0 tangle is two non-crossing horizontal strings. These are shown in Figure 13. One constructs other tangles by starting with the ∞ or 0 tangle and twisting the strings around each other a certain number of times. There are two different types of twisting involved in

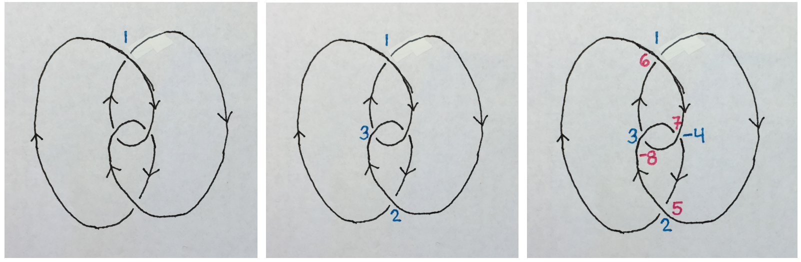


Figure 9: (a) Label an undercrossing with a 1. (b) Label crossings in order, following the orientation. (c) Finish labeling the crossings.

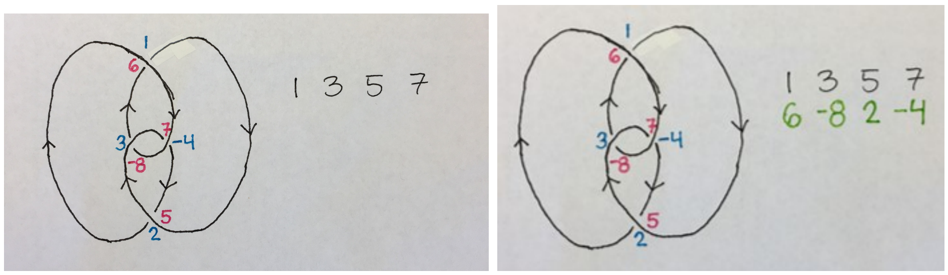


Figure 10: (a) Write down the odd numbers in order. (b) The Dowker Notation for this projection of the Figure Eight knot

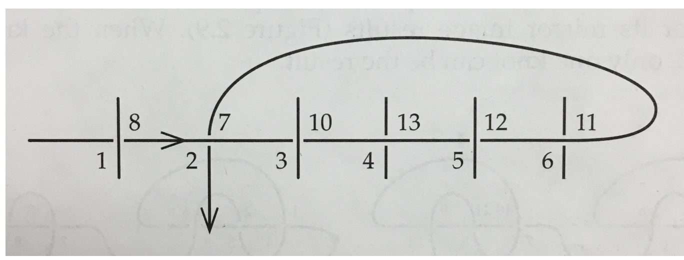


Figure 11: Beginning to redraw the knot with Dowker notation 8 10 12 2 14 6 4 [1]

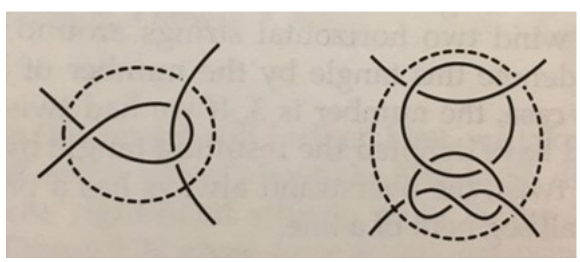


Figure 12: Two tangles [1]

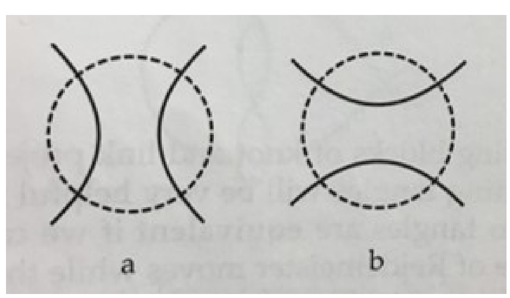


Figure 13: (a) The ∞ tangle (b) The 0 tangle [1]

creating tangles. For a positive integer twist, the overstrand has positive slope, while for a negative integer twist, the overstrand has negative slope. This is shown in Figure 14. **Rational tangles** are

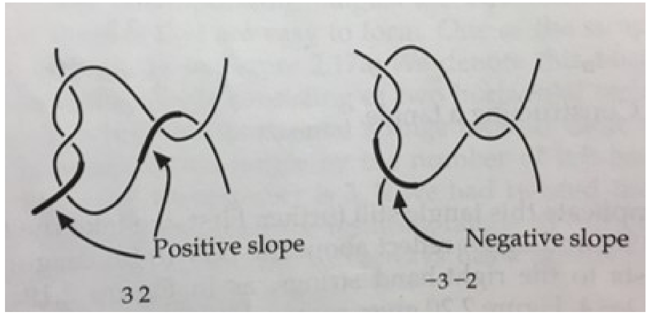


Figure 14: Positive and Negative integer twists [1]

constructed by taking the first tangle, rotating it 90 degrees clockwise, switching all the crossings in order to maintain their positive or negative character, then twisting the two strands on the right into another tangle. An example of this is given in Figure 15.

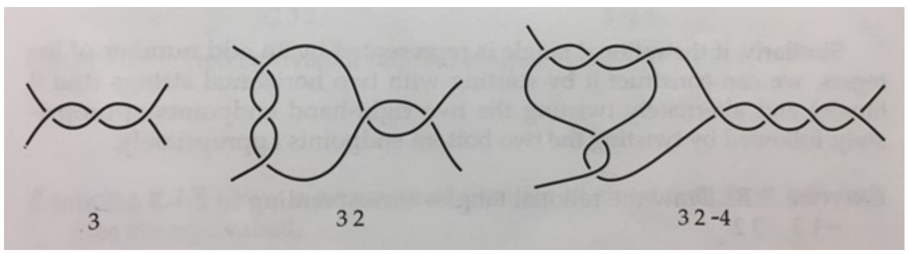


Figure 15: Examples of rational tangles [1]

The Conway notation for knots made of rational tangles involves the order in which the tangles were created and whether they were made of positive or negative integer twists. For instance, the knot with Conway notation 3 1 -2 was made of three positive integer twists, followed by one positive integer twist, followed by two negative integer twists. Additionally, the Conway notation for the trefoil knot is 3, as it is made of three positive integer twists, while the Conway notation for the figure eight knot is 2 2, as it is made of two positive integer twists followed by another two positive integer twists.

2.5 Knot Invariants

A **knot invariant** is a quantity that one can calculate from a knot that doesn't change when the knot is moved in space. In other words, it is a quantity that may help distinguish knots from each other. Some knot invariants are (double) crossing number, triple crossing number, and unknotting number.

2.5.1 Crossing Number

The **(double) crossing number** of a knot is the least number of (double) crossings in any projection of the knot, denoted $c(K)$ for a knot K. In this case, a (double) crossing is defined as the point at which two strands of the knot meet, hence the name of double crossing. A projection of a knot is **reduced** when there are no easily removed crossings. Figure 16 (a) shows the types of crossings that make a projection not reduced, while Figure 16 (b) shows projections that are not reduced. A projection is **alternating** if while traversing the knot, the crossings switch between overcrossings and undercrossings. Figure 17 shows an alternating knot. Unfortunately, there is no systematic way to determine the crossing number of a knot. However, it has been proven that an alternating knot in a reduced alternating projection of n crossings has crossing number n; this fact was proven in [4].

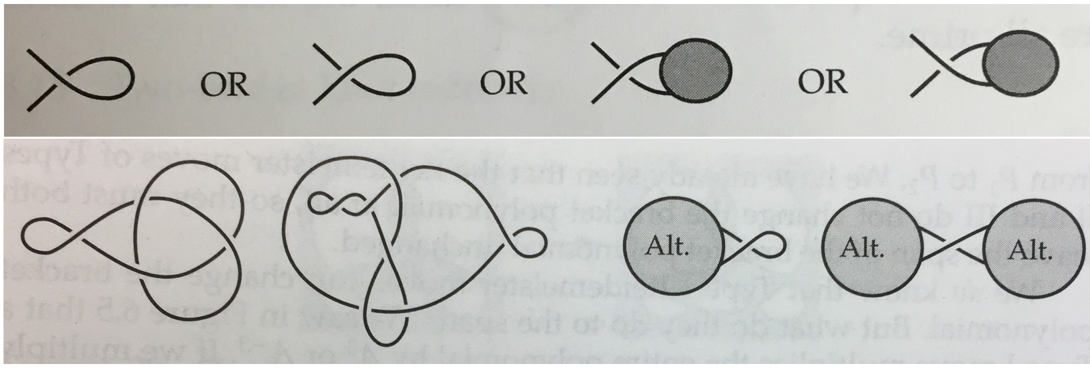


Figure 16: (a) Reducible Crossings in a Projection [1]
(b) Projections with Reducible Crossings; Alt. stands for alternating [1]

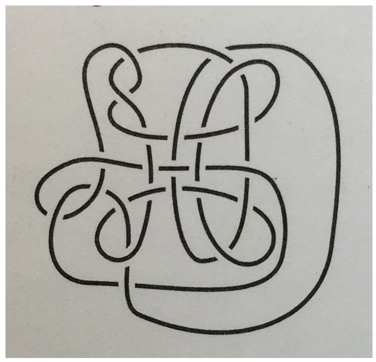


Figure 17: An alternating knot with 23 crossings [1]

2.5.2 Triple Crossing Number

Triple crossing number is similar to double crossing number, with the exception that one is counting triple crossings. A **triple crossing** is defined as the point at which three strands of the knot meet; as shown in Figure 18, the strands are labeled T for top, M for middle, and B for bottom. A **triple-crossing projection** is a projection of a knot with exclusively triple crossings. Hence, the **triple crossing number** of a knot is the minimum number of crossings in any triple-crossing projection of a knot. The triple crossing number of a knot K is denoted $c_3(K)$. Figure 19 shows

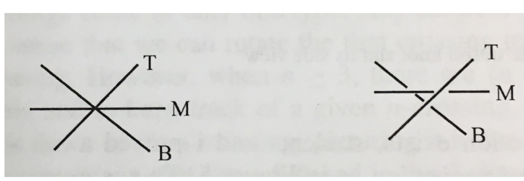


Figure 18: A triple crossing from the top and slightly to the side [2]

triple crossing projections of the trefoil and figure 8 knots.

In [2], Adams describes a process for turning double crossings into triple crossings, which will be summarized here. Letting K be a double crossing projection of a knot, define a **crossing covering circle**, denoted C, to be a topological circle intersecting K only at crossings; specifically, where C does intersect K, there are two strands of K coming out either side of C. This is shown in Figure 20 (a). A **crossing covering collection** is a set of disjoint crossing covering circles in which every crossing in K intersects exactly one of the circles; in other words, a crossing covering collection is made up of enough circles to cover all the crossings of K. A crossing covering collection for the figure 8 knot is shown in Figure 20 (b). Once a crossing covering circle is found for K, one of the

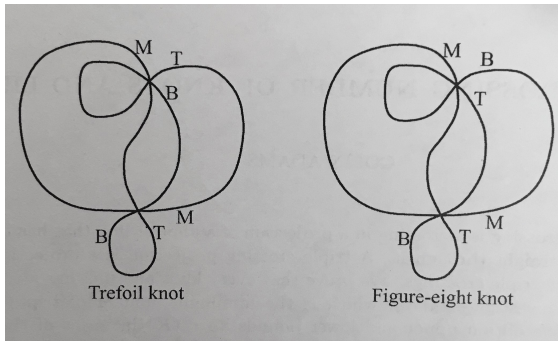


Figure 19: Triple crossing projections [2]

strands from a crossing can be stretched around the crossing covering circle, so that it lays on top of each double crossing that the circle intersects, as shown in Figures 21 (a) and 21 (b). One can then perform a Type I Reidemeister move to eliminate the loop formed, as in Figure 21 (c), then perform a Type II Reidemeister move to eliminate the double crossings formed, as in Figure 21 (d). Hence, these two Reidemeister moves reduce the number of triple crossings by 1 and effectively split the crossing. Ergo, if the crossing covering circle initially intersected n crossings, the result is $n-1$ triple crossings in the new projection. The operation described here and shown in Figure 21 is called a **folding**.

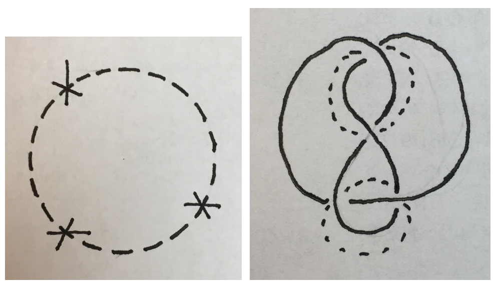


Figure 20: (a) A crossing covering circle. (b) A crossing covering collection for the figure 8 knot

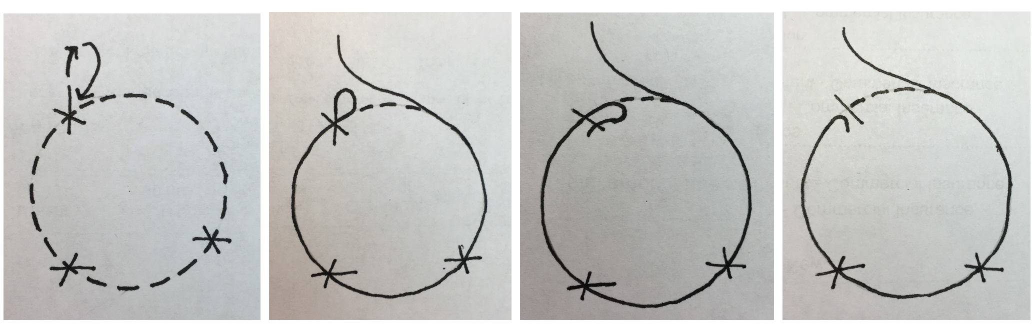
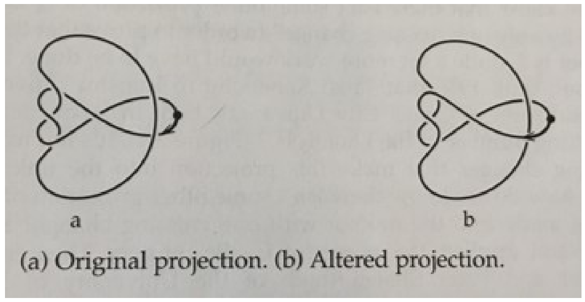


Figure 21: (a) Stretch a strand up and around the crossing covering circle. (b) A strand stretched around the crossing covering circle. (c) Undo the loop with a Type I Reidemeister move. (d) Undo the double crossings with a Type II Reidemeister move.

2.5.3 Unknotting Number

The **unknotting number** of a knot is the minimum number of crossings in any projection of the knot that must be changed in order to make the knot into a projection of the unknot. A knot can be unknotted by traversing the knot in a certain direction and changing all the undercrossings to overcrossings; if the crossing has already been traversed, then leave it as an undercrossing. Figure 22 shows this process. While this method will not always produce the minimum unknotting number,



(a) Original projection. (b) Altered projection.

Figure 22: The unknotting process [1]

it will change a projection of a knot into a projection of the unknot. Surprisingly, the unknotting number for the knot is not always found from a projection with the crossing number. For instance, the knots with Conway notation 5 1 4 and 2 -2 2 -2 2 4 are equivalent; the first projection has

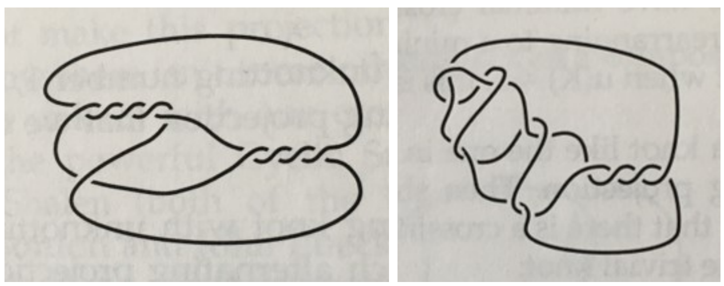


Figure 23: (a) The knot 5 1 4 [1] (b) The knot 2 -2 2 -2 2 4 [1]

ten crossings, as shown in Figure 23 (a), while the second has fourteen, as shown in Figure 23 (b). The projection 5 1 4 requires three crossings to be switched in order to unknot it, but the second projection only requires switching two crossings. Ergo, the unknotting number for that knot is two, but it was obtained from a projection with more crossings than the crossing number.

2.6 Braid Representation of a Knot

While a braid is not a certain type of knot nor a knot invariant, every knot can be represented as a braid, and different properties can be observed from the braid representation of a knot. An **open braid** is a set of strings attached at both ends to a horizontal bar; the strings then weave over and under each other. Figure 24 (a) shows an open braid. A **closed braid** is obtained by pulling

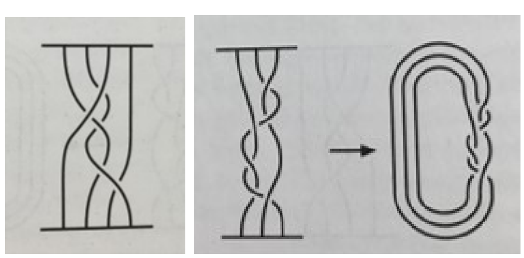


Figure 24: (a) An open braid [1] (b) The closure of a braid [1]

the two bars around in an oval and gluing them together, creating a knot or link. Figure 24 (b) shows an open braid and its closed braid representation. The **braid index** of a knot or link is the minimum number of strings in any open-braid representation of the knot or link. For instance, the braid in Figure 24 (a) has braid index 4, whereas the braid in Figure 24 (b) has braid index 3.

2.6.1 Braid Notation

To describe a braid, one lists the crossings in order. If the first string crosses over the second string, it is denoted by σ_1; if the first string crosses *under* the second, it is denoted by σ_1^{-1}. These are shown in Figure 25. More generally, if the i^{th} string crosses over the $i+1^{st}$ string, it is denoted by

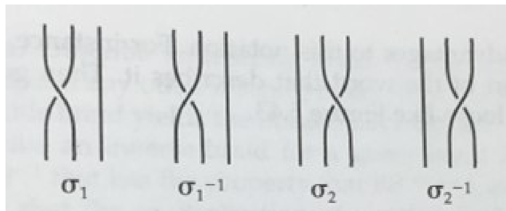


Figure 25: Braid crossings [1]

σ_i; if the i^{th} string crosses under the $i+1^{st}$ string, it is denoted by σ_i^{-1}. Attaching these letters together forms a **word** that describes the braid. For example, a word to describe a five-string braid could be $\sigma_1\sigma_2^{-1}\sigma_3\sigma_4^{-1}\sigma_3^2\sigma_2\sigma_1^{-1}$. Additionally, if a braid's word is $\sigma_1\sigma_1\sigma_1\sigma_1\sigma_1$, it is abbreviated σ_1^5. Similarly, if a braid's word is $\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_3$, it is abbreviated $(\sigma_1\sigma_2\sigma_3)^2$.

3 Problem

This section introduces the background information pertaining to the category of knot that is the subject of the problem investigated in Section 4. Accordingly, the torus itself and torus knots are introduced, multiple notations for torus knots are given, and patterns amongst the invariants of torus knots are explored.

3.1 Torus Knots

A **surface** is the outside layer of an object, such as the glaze of a doughnut or the plastic of a beach ball. For instance, the surface of a beach ball is a **sphere**, while the surface of a doughnut is called a **torus**. These two surfaces are shown in Figure 26. **Torus knots** are knots that can lie

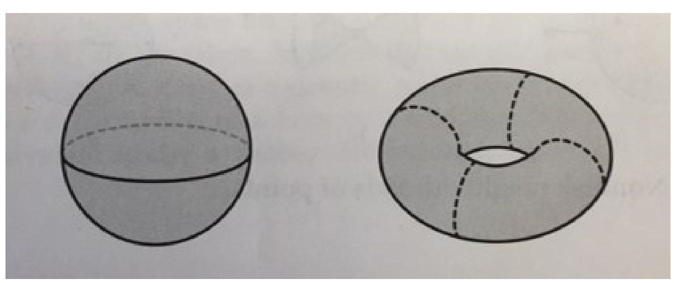


Figure 26: A sphere and a torus [1]

on an unknotted torus without crossing over or under themselves on the torus. This means that anywhere there would be a crossing in a two dimensional projection, the overstrand lies on top of the torus and the understrand lies underneath the torus; if the torus is transparent, a crossing can still be seen from above. Figure 27 shows two torus knots. A curve that runs around a torus the short way is called a **meridian curve**, and a curve that runs around a torus the long way is called a **longitude curve**. These are shown in Figure 28 (a). The **core curve** of a torus is the longitude curve that runs through the center of each meridian curve, as shown in Figure 28 (b); the core curve defines the shape of the torus.

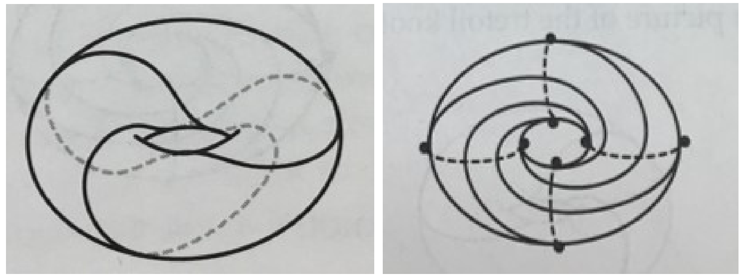


Figure 27: Two torus knots [1]

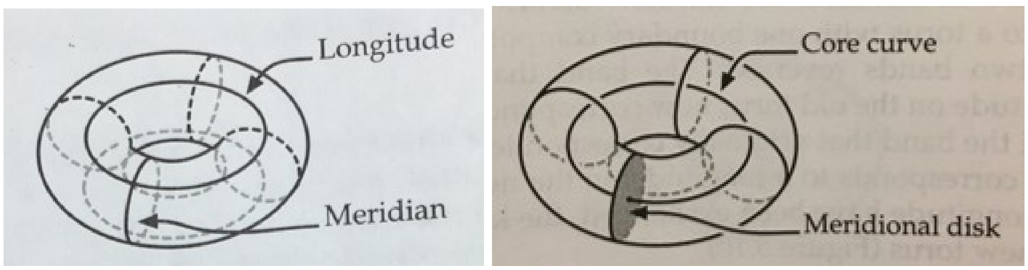


Figure 28: (a) Meridian and longitude curves [1] (b) The core curve of a torus [1]

3.1.1 Notation for Torus Knots

A knot is a (p, q)**-torus knot** if it wraps around the torus p times meridionally and q times longitudinally when p and q are relatively prime. If p and q are not relatively prime, it will be a (p, q)-torus link, not a knot. For example, the trefoil knot is a (3, 2)-torus knot, as shown in Figure 29. Conveniently, every (p, q)-torus knot is also a (q, p)-torus knot. [1]

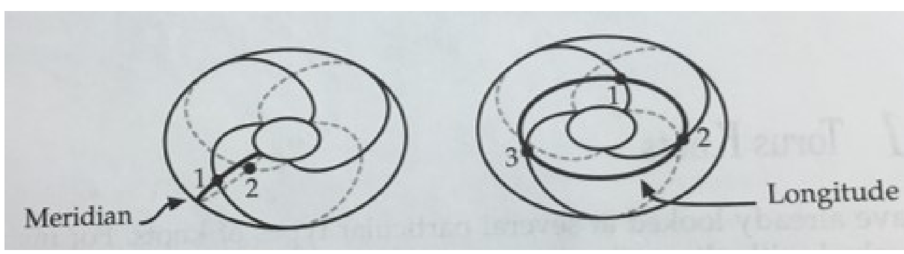


Figure 29: The trefoil is a (3,2)-torus knot [1]

3.1.2 Dowker Notation of Torus Knots

There are some interesting patterns of the Dowker notation for torus knots. For the $(p, 2)$-torus knots, the Dowker notation is entirely positive, which means it is an alternating knot; it starts at $p + 1$, increases until $2p$, then restarts increasing from 2 until $p - 1$. For the $(p, 3)$-torus knots, every other number is negative (meaning that the knot is not alternating); it increases until $\pm 4p$, then restarts increasing from 2. For the $(p, 4)$-torus knots, the most noticeable pattern is that the sign of the numbers follows the pattern $+--+--$; similarly, for the $(p, 5)$-torus knots, the most noticeable pattern is that the sign of the numbers follows the pattern $++--++--$.

3.1.3 Conway Notation of Torus Knots

The Conway notation for the $(p, 2)$-torus knots is simply p, as a $(p, 2)$-torus knot is two strands twisted around each other p times. The Conway notation for the $(4, 3)$-torus knot is $3, 3, 2-$, while the Conway notation for the $(5, 3)$-torus knot is $5, 3, 2-$. However, for other torus knots the pattern for the Conway notation is not simple and not needed for the scope of this paper.

3.1.4 Crossing Number and Torus Knots

Fortunately, there is a distinct pattern in the crossing number of torus knots. For a (p, q)-torus knot with $p > q$, the crossing number is $p(q - 1)$. This means that for a $(p, 2)$-torus knot, the crossing number is p; for a $(p, 3)$-torus knot, the crossing number is $2p$; for a $(p, 4)$-torus knot, the crossing number is $3p$; for a $(p, 5)$-torus knot, the crossing number is $4p$, and so on. Additionally, note that the crossing number for a $(p, 2)$-torus knot is odd, since p and 2 must be coprime.

3.1.5 Unknotting Number and Torus Knots

As with crossing number, there is a pattern in the unknotting number of torus knots. For a (p, q)-torus knot, the unknotting number is $\frac{1}{2}(p - 1)(q - 1)$. This means that for a $(p, 2)$-torus knot, the unknotting number is $\frac{1}{2}(p - 1)$; for a $(p, 3)$-torus knot, the unknotting number is $p - 1$; for a $(p, 4)$-torus knot, the unknotting number is $\frac{3}{2}(p-1)$; for a $(p, 5)$-torus knot, the unknotting number is $2(p - 1)$, and so on.

3.1.6 Braid Representation of a Torus Knot

For torus knots represented as braids, there are patterns in both the braid indices and the braid words. For a (p, q)-torus knot with $p > q$, the braid index is simply q, meaning that a $(p, 2)$-torus knot has braid index 2, a $(p, 3)$-torus knot has braid index 3, $(p, 4)$-torus knot has braid index 4, a $(p, 5)$-torus knot has braid index 5, and so on. Braid words for torus knots have a more interesting pattern. A $(p, 2)$-torus knot has the word σ_1^p; a $(p, 3)$-torus knot has the word $(\sigma_1\sigma_2)^p$; a $(p, 4)$-torus knot has the word $(\sigma_1\sigma_2\sigma_3)^p$; and a $(p, 5)$-torus knot has the word $(\sigma_1\sigma_2\sigma_3\sigma_4)^p$. In general, the pattern for the braid word of a (p, q)-torus knot with $p > q$ is $(\sigma_1 \ldots \sigma_{q-1})^p$.

3.2 Triple Crossing Number of Torus Knots

In [2], Adams proposes as Theorem 2.3 and proves the following statement:

> Every knot or link K other than a 2-braid knot has a triple crossing projection with no more than $c(K) - 2$ triple crossings, and a 2-braid knot has a triple crossing projection with no more than $c(K) - 1$ triple crossings.

Since the 2-braid knots are just the $(p, 2)$-torus knots, and the double crossing number of a $(p, 2)$-torus knot is p, the last part of the theorem becomes $c_3((p, 2)\text{-torus knot}) \leq p - 1$. Referring back to section 2.6.2, if a crossing covering circle intersects n double crossings, then a folding creates $n - 1$ triple crossings. This means that a folding decreases the relative crossing number by 1 for each crossing covering circle, so $c_3(K) = c(K) - d$, where d is the number of crossing covering circles in the maximal crossing covering collection. Here, the maximal crossing covering collection, or the crossing covering collection that has the most possible crossing covering circles, is needed so that $c(K) - d$ is minimal; since the triple crossing number is defined as the minimum number of crossings in any triple-crossing projection of a knot, $c(K) - d$ being minimal ensures that the triple crossing number has been realized. Hence, the theorem implies that any knot or link other than a 2-braid knot has a minimum of 2 crossing covering circles in its maximal crossing covering collection, while a 2-braid knot has a minimum of 1 crossing covering circle in its maximal crossing covering collection.

4 Results

In order to prove the desired result, two lemmas are necessary.

Lemma 1. *If a crossing covering circle intersects only one crossing of a knot, then the crossing must be trivial.*

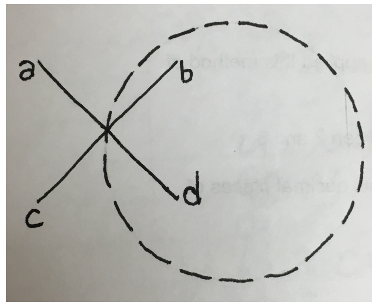


Figure 30: A crossing covering circle intersecting only one crossing

Proof. Suppose a crossing covering circle only intersects one crossing, as in Figure 30. Then the four strands coming out of the crossing can be connected in 6 ways: a with b, c with d, a with c, b with d, a with d, and b with c. First, if a were to connect with b, part of the knot would have to intersect the crossing covering circle, which could happen at either a crossing or not at a crossing. If the intersection is at another crossing, then the hypothesis that this crossing covering circle intersects only one crossing has been contradicted; if the intersection is not at a crossing, then the definition of a crossing covering circle is contradicted. Hence, it is not possible to connect a with b; the argument for connecting c with d is similar. Second, if b were to connect with c, not only would the same problems occur as in the a with b case, but since b and c are already on the same strand, connecting these two ends without passing through a or d first would create a component of a link, which contradicts the assumption that this is a knot. Hence, it is not possible to connect b with c; the argument for connecting a with d is similar.

Third, if a were to connect with c, either a simple loop would be formed (no added crossings), or there would be at least one additional crossing between a and c. In the first case, the loop formed is exactly the type of loop that can be created or removed with a Type I Reidemeister move. In the second case, the crossing can also be undone with a Type I Reidemeister move, which has the effect of flipping the crossings between a and c upside-down in the projection plane, but not fundamentally changing the knot. Moreover, this Type I Reidemeister move will not create unwanted crossings elsewhere in the knot, as that would require either b or d to cross the strand between a and c, which would in turn contradict either the definition of a crossing covering circle or the hypothesis, as explained earlier. Ergo, in either case, the crossing can be removed with a Type I Reidemeister move without fundamentally changing the knot. Note that supposing a connects with c implies b connects with d, since otherwise either the definition of a crossing covering circle or the hypothesis would be contradicted, as explained earlier.

In short, the only valid way for a crossing covering circle to intersect only one crossing is for a to connect with c and b to connect with d. This crossing can be removed with a Type I Reidemeister move without fundamentally changing the knot. Since Type I Reidemeister moves deal exclusively with trivial crossings, the one crossing intersecting the crossing covering circle must be trivial. ■

Lemma 2. *The maximal crossing covering collection for the trefoil knot contains one crossing covering circle.*

Proof. To find the maximal crossing covering collection for the trefoil, it is natural to start by trying to find a crossing covering collection with one crossing covering circle, then with two circles, then with three. Since the trefoil has three double crossings, it is impossible for the trefoil to have a valid crossing covering collection with four or more crossing covering circles; due to the pigeonhole principle, the crossing covering circles would not be disjoint, which contradicts the definition of a crossing covering collection.

To begin, the simplest choice for the first crossing covering circle is the one that covers all three crossings, as shown in Figure 31 (a); since it crosses the trefoil only at crossings, and two strands of the knot are on either side of the circle at each crossing, this is a valid crossing covering circle. Moreover, since this crossing covering circle covers all three double crossings of the trefoil, it is also a crossing covering collection. But are there more crossing covering circles in the maximal crossing covering collection of the trefoil? Accordingly, consider the circle in Figure 31 (b); it is also a valid

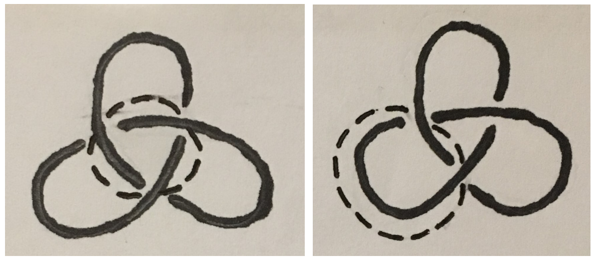


Figure 31: Two crossing covering circles for the trefoil

crossing covering circle, since it crosses the trefoil only at crossings and two strands of the knot are on either side of the circle at each crossing. However, this leaves one crossing uncovered. There are two ways to cover this last crossing, as shown in Figure 32. Since a crossing covering circle must

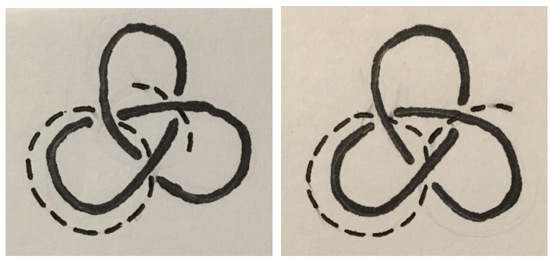


Figure 32: (a) One option for a second crossing covering circle. (b) Another option for a second crossing covering circle.

intersect the knot only at crossings and have two strands of the knot coming out either side at each crossing, the circle in Figure 32 (a) must continue on to cover the other two crossings in order to be a valid crossing covering circle, as shown in Figure 33 (a). However, the crossing covering circles in

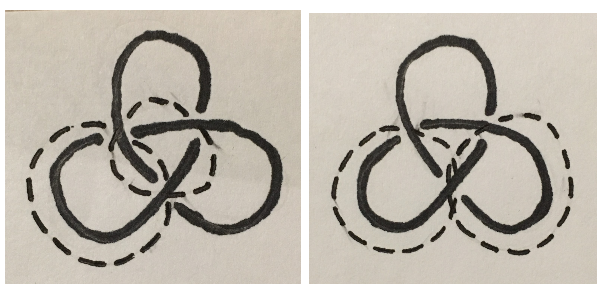


Figure 33: The second crossing covering circle corresponding to Figure 32 (a). (b) The second crossing covering circle corresponding to Figure 32 (b).

a crossing covering collection must be disjoint, so the two crossing covering circles shown in Figure 33 (a) do not form a valid crossing covering collection. On the other hand, by the definition of a crossing covering circle, the circle in Figure 32 (b) must continue on to cover one of the other two crossings in order to be a valid crossing covering circle, as shown in Figure 33 (b). However, since the crossing covering circles in a crossing covering collection must be disjoint, the two crossing covering circles shown in Figure 33 (b) do not form a valid crossing covering collection. Thus, there does not exist a crossing covering collection for the trefoil with two crossing covering circles.

Even if the trefoil does not have a crossing covering collection with two circles, might it have a crossing covering collection with three circles? Accordingly, there is only one way that three crossing covering circles could exist for the trefoil such that the circles in the collection are disjoint. This is for each crossing to have its own crossing covering circle; hence, each crossing covering circle has one crossing. However, by Lemma 1, each crossing in the trefoil knot must be trivial. Since the trefoil is not equivalent to the unknot, this is a contradiction; as such, it is impossible for each crossing to have its own crossing covering circle. Ergo, there is no crossing covering collection for the trefoil with three crossing covering circles.

Consequently, the only valid crossing covering collection for the trefoil knot is the one in Figure 31 that consists of only one crossing covering circle. Hence, the maximal crossing covering collection for the trefoil knot has exactly one crossing covering circle. ■

With the previous lemmas established, the following theorem can now be proved.

Theorem 1. *K is a $(p, 2)$-torus knot if and only if $c_3(K) = c(K) - 1$.*

Proof. ($\Rightarrow$) The principle of mathematical induction is utilized to prove the theorem. As such, consider the torus knots of the form $(2n+1, 2)$; since $2n+1$ is odd $\forall n \in \mathbb{N}$, $2n+1$ and 2 are always coprime. Hence, the $(2n+1, 2)$-torus knot will indeed be a torus *knot* and not a torus *link*.

Base case: $n = 1$ Since $2(1) + 1 = 3$, analyzing the case where $n = 1$ is equivalent to analyzing the $(3, 2)$-torus knot, which is the trefoil knot. To determine the triple crossing number of the trefoil, the maximal crossing covering collection is needed. By Lemma 2, the maximal crossing covering collection of the trefoil has one crossing covering circle. Thus, the triple crossing number for the trefoil can now be found by performing a folding using the maximal crossing covering collection. Figure 34 (a) shows the maximal crossing covering collection for the trefoil. In Figure 34 (b), a

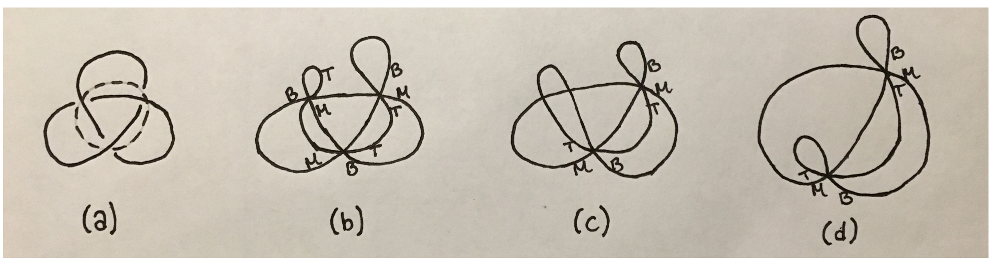


Figure 34: Performing a folding on the trefoil

strand of the trefoil is stretched around the crossing covering circle so that it lays on top of all the crossings. Figure 34 (c) shows the effect of a Type I Reidemeister move on the left loop, while Figure 34 (d) shows a Type II Reidemeister move on the same loop. With this sequence of moves, the trefoil has been folded to produce a triple crossing projection that realizes the triple crossing number of the trefoil. As such, Figure 34 (d) shows that the triple crossing number of the trefoil is 2. Since the double crossing number of the trefoil is 3, and $2 = 3 - 1$, it can be concluded that $c_3(K) = c(K) - 1$ for the $(3, 2)$-torus knot; that is, $c_3((2n+1, 2)\text{-torus knot}) = c((2n+1, 2)\text{-torus knot}) - 1$ for $n = 1$.

Inductive step: Suppose $c_3((2n+1, 2)\text{-torus knot}) = c((2n+1, 2)\text{-torus knot}) - 1 \quad \forall n \leq k$. Now consider the case where $n = k+1$; that is, consider $c_3((2(k+1)+1, 2)\text{-torus knot}) = c_3((2k+3, 2)\text{-torus knot})$. Since the double crossing number of a $(p, 2)$-torus knot is p, $c((2k+1, 2)\text{-torus knot}) = 2k+1$ and $c((2k+3, 2)\text{-torus knot}) = 2k+3$. Since $(2k+3) - (2k+1) = 2$, the $(2k+3, 2)$-torus knot has two more double crossings than the $(2k+1, 2)$-torus knot. As such, the $(2k+3, 2)$-torus knot can be created from the $(2k+1, 2)$-torus knot as shown in Figure 35. Since $c_3((2n+1, 2)$-torus

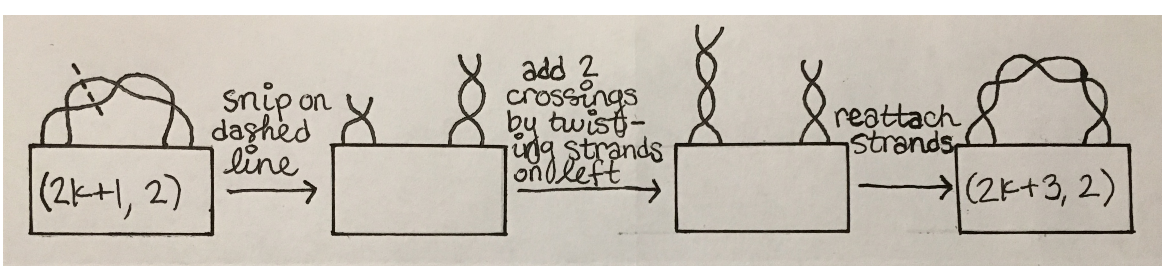


Figure 35: Creating the $(2k+3, 2)$-torus knot from the $(2k+1, 2)$-torus knot

knot) $= c((2n+1,2)$-torus knot$) - 1 \quad \forall n \leq k$, there is exactly one crossing covering circle in the maximal crossing covering collection of a $(2n+1,2)$-torus knot $\quad \forall n \leq k$. Since the $(2k+3,2)$-torus knot has two more double crossings than the $(2k+1,2)$-torus knot, these two double crossings could effect the maximal crossing covering collection in one of three ways:

Case 1: Each of the two crossings is added to a different crossing covering circle. However, this requires requires one of two things: that two crossing covering circles exist in the maximal crossing covering collection of the $(2k+1,2)$-torus knot, so that each crossing can be added to a separate crossing covering circle, or that one crossing is added to an existing crossing covering circle and the other crossing is added to a new crossing covering circle. For the first option, $\forall n \leq k$, the maximal crossing covering collection of a $(2n+1,2)$-torus knot has exactly one crossing covering circle, so it is impossible for the new crossings to be added to two existing crossing covering circles. For the second option, it is certainly possible for one crossing to be added to the existing crossing covering circle; however, by Lemma 1, if a crossing covering circle only has one crossing, the crossing must be trivial, so the second added crossing would be trivial, in which case the knot would not be a $(2k+3,2)$-torus knot. Hence, case 1 is impossible.

Case 2: The crossings form their own new crossing covering circle, thereby increasing the number of crossing covering circles in the maximal crossing covering collection to 2. Accordingly, consider the crossing covering circles in Figure 36; Figure 36 (a) shows the crossing covering circle for the $(2k+1,2)$-torus knot, while Figure 36 (b) shows two crossing covering circles for the $(2k+3,2)$-torus knot: one that covers $2k+1$ crossings, and one that covers the two new crossings. Since the

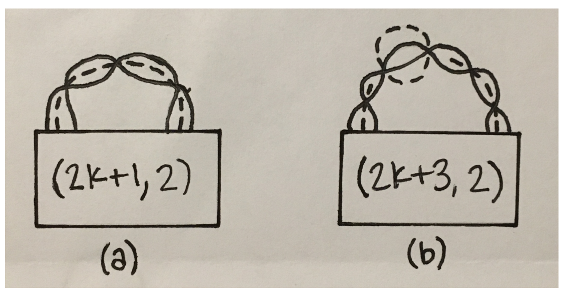


Figure 36: Crossing covering circles for the $(2k+1,2)$-torus knot and the $(2k+3,2)$-torus knot

crossing covering circles in a crossing covering collection must be disjoint, the only ways these two crossing covering circles do not intersect are for the crossing covering circle covering $2k+1$ crossings to avoid the other crossing covering circle either inside or outside the torus knot. However, both the inside and outside options require the crossing covering circle to either cross through the knot *not* at a crossing, or to pass through the knot at a crossing with three strands on one side and one on the other, which both contradict the definition of a crossing covering circle. Hence, two disjoint crossing covering circles cannot be created for the $(2k+3,2)$-torus knot in this way. These cases are shown in Figure 37.

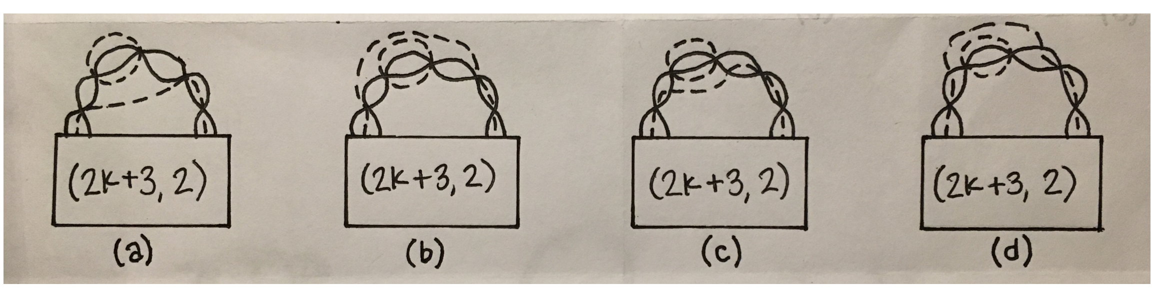


Figure 37: In (a) & (c), the crossing covering circle connects inside the torus knot; in (b) & (d), the crossing covering circle connects outside the torus knot; in (a) & (b), the crossing covering circle passes through a crossing with three strands on one side; in (c) & (d), the crossing covering circle passes through the knot not at a crossing

Another way to have two disjoint crossing covering circles is to leave the crossing covering circle that covers $2k+1$ crossings intact and then add two double crossings, as in Figure 38. However, in

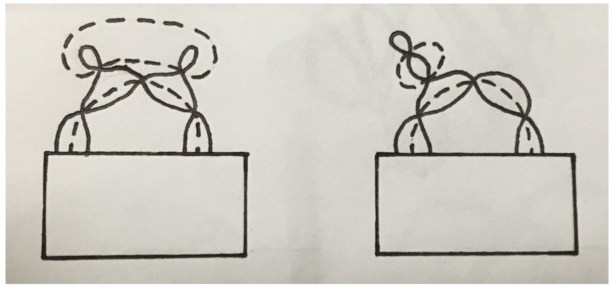


Figure 38: Two more ways to have two disjoint crossing covering circles

both cases, the added double crossings are trivial and can be reduced by a Type I Reidemeister move; hence, the knot is still a $(2k+1,2)$-torus knot with one crossing covering circle in its maximal crossing covering collection. Furthermore, there could exist a crossing covering collection with three crossing covering circles if each of the two new crossings were added to their own new crossing covering circle. However, both of the new crossing covering circles would only intersect one crossing, so by Lemma 1, both new crossings are trivial; thus, the knot is still a $(2k+1,2)$-torus knot. Consequently, case 2 is impossible.

Case 3: Both crossings are added to the one crossing covering circle existing in the maximal crossing covering collection of the $(2k+1,2)$-torus knot, thereby not increasing the number of crossing covering circles in the maximal crossing covering collection. This is shown in Figure 39. Since cases 1 and 2 are impossible, case 3 must be true. Accordingly, the $(2k+3,2)$-torus knot has exactly one crossing covering circle in its maximal crossing covering collection, so $c_3((2k+3,2)$-torus knot$) = c((2k+3,2)$-torus knot$) - 1$.

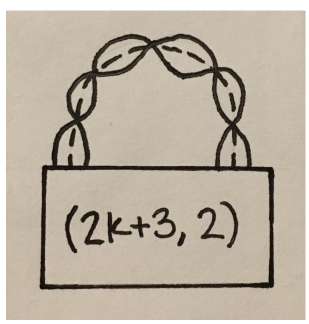


Figure 39: The one crossing covering circle for the $(2k+3,2)$-torus knot

By the principle of mathematical induction, $c_3((2n+1,2)$-torus knot$) = c((2n+1,2)$-torus knot$) - 1 \quad \forall n \in \mathbb{N}$.

($\Leftarrow$) Suppose $c_3(K) = c(K) - 1$ for a knot K. By theorem 2.3 of [2] (which is quoted in section 3.2), if K is a $(p,2)$-torus knot, then $c_3(K) \leq c(K) - 1$; if K is not a $(p,2)$-torus knot, then $c_3(K) \leq c(K) - 2$. Now suppose K is not a $(p,2)$-torus knot. Then $c(K) - 1 = c_3(K) \leq c(K) - 2$, which is equivalent to $c(K) - 1 \leq c(K) - 2$. Adding 1 to both sides gives $c(K) \leq c(K) - 1$. Since an integer cannot be less than or equal to one less than itself, a contradiction has been reached. Ergo, K must be a $(p,2)$-torus knot. ■

5 Conclusion and Future Research

This paper has introduced knots and links along with their various notations. Some important knot invariants, such as double and triple crossing number, were explained. Torus knots were introduced and patterns in the invariants of torus knots were explored. Specifically, the triple crossing number of the $(p, 2)$-torus knots was investigated; for a $(p, 2)$-torus knot, the triple crossing number is equal to the double crossing number minus one. This implies that the $(p, 2)$-torus knots are the only knots that have a maximal crossing covering collection with exactly one crossing covering circle; all other knots and links, as proved in Adams's Theorem 2.3 [2], have to have at least two crossing covering circles in their maximal crossing covering collection.

There are quite a few questions left to research in this section of knot theory. For instance, is there a specific family of knots for which $c_3(K) = c(K) - 2$ holds? What about $c_3(K) = c(K) - n$ for some $n > 2$? Are there commonalities amongst the knots for which $c_3(K) = c(K) - n$ holds for the same n? These questions are aimed at classifying knots by their triple crossing number relative to their double crossing number. This then begs the question, for knots with the same triple crossing number, how do the values of n compare? How do the double crossing numbers compare? However, these questions depend on knowing the triple crossing numbers for more knots than the triple crossing numbers are known for; as of 2017 [3], the triple crossing numbers for most of the knots with 8 or more crossings are unknown. Specifically, 39 of the 70 prime knots with 8 or 9 double crossings have unknown triple crossing numbers, so there is still work to be done before triple crossing number can be analyzed comprehensively.

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